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On the 2nd obstruction on the existence of the Toda-Smith $V(4)$

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§ 1. Introduction

Let BP be the Brown-Peterson spectrum at a prime p. The Toda-Smith spectrum $V(n)$ is a spectrum with BP_*-homology BP_*/I_{n+1} for the ideal $I_n = (p, v_1, \cdots, v_{n-1})$ of the polynomial algebra $BP_* = \boldsymbol{Z}[v_1, v_2, \cdots]$. To construct the spectra $V(n)$'s, we adopt the method using the Adams-Novikov spectral sequence based on BP, which says: Suppose that there exists a ring spectrum $V(n-1)$. Then compute the E_2-term to see whether or not the targets of the differentials $d_r v_n$ $(r \geq 2)$ of v_n are trivial. If the differentials are all trivial, then we have an element v_n of $\pi_* V(n-1)$ whose BP_*-Hurewicz image is v_n. Composite the multiplication of $V(n-1)$ and the map $V(n-1) \wedge v_n$, and we get the self-map $v(n)$ of $V(n-1)$, whose cofiber is $V(n)$. Due to Toda's computation [8], we read off that the differentials are all trivial if $n \leq 3$ and $p \geq 2n+1$, and we have five obstructions

$$b_{2,0}^{p-3} l_4 k_0,\ b_{1,0}^{p-2} b_{1,1}^{p-2} b_{2,0} l_1,\ b_{1,0}^{p^2-2p-2} b_{2,0}^3 h_1,$$
$$b_{1,0}^{p^2-2p-2} b_{2,0} k_1 l_1 \quad \text{and} \quad b_{1,0}^{p^2-p-2} b_{2,0} l_2$$

if $n = 4$ and $p \geq 9$ in the E_2-term $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_4)$ of the Adams-Novikov spectral sequence for $V(3)$. The first obstruction was shown to be no longer an obstruction since we see that $d_{2p-1} v_4 = 0$ in [6]. In this note we show that the second one is also trivial, which implies

THEOREM. *Let p be a prime greater than* 9 *and r non-negative integer with $r < 2p^2 - 4p + 3$. Then*

$$d_r v_4 = 0$$

is the Adams-Novikov spectral sequence for $V(3)$.

We use some results given in [6] to show an element trivial.

§ 2. Homotopy groups of finite spectra

Let p be a prime greater than 7 and BP denote the Brown-Peterson spectrum at p. Then $BP_* = \boldsymbol{Z}_{(p)}[v_1, v_2, \cdots]$ for Hazewinkel's generators v_i's with $|v_n| = 2p^n$

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-2. The Toda-Smith spectrum $V(3)$ is defined to be $BP_* V(3) = BP_*/I_4$ for the prime ideal $I_n = (p, v_1, \cdots, v_{n-1})$ and the Ravenel spectrum $T(1)$ to be $BP_* T(1) = BP_*[t_1]$ as a subcomodule algebra of $BP_*BP = BP_*[t_1, t_2, \cdots]$. Here $|t_n| = 2p^n - 2$. Let $T(1)^{(a)}$ denote the a-skeleton of $T(1)$ and consider a finite spectrum $X(k) = V(3) \wedge T(1)^{(kq)}$ for $q = 2p - 2$, and we have

$$BP_* X(k) = (BP_*/I_4)\{1, t_1, \cdots, t_1^k\}.$$

For the Brown-Peterson spectrum BP and a p-local connected spectrum X, we have the Adams-Novikov spectral sequence converging to the homotopy group $\pi_* X$ with E_2-term

$$E^{s,t}X = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*X).$$

We note here that this E_2-term for $X = X(k)$ is isomorphic to

$$F^{s,t}X(k) = \mathrm{Ext}^{s,t}_{P(3)}(F_p, M(k))$$

for $t - s < 2p^4 - 2$ by degree reason, where

$$P(k) = F_p[t_1, t_2, \cdots, t_k] \text{ and } M(k) = F_p\{1, t_1, t_1^2, \cdots, t_1^k\}.$$

Thus we get some information on the homotopy group $\pi_r X(p-1)$ for $r < 2p^4 - 2$ by studying $F^{s,t}X(p-1)$. We have the Cartan-Eilenberg spectral sequences

$$\begin{aligned}(2.1)\qquad &\mathrm{Ext}_{P_2}(F_p, F_p) \otimes \mathrm{Ext}_{P_1}(F_p, M(p-1)) \Rightarrow \mathrm{Ext}_{P(2)}(F_p, M(p-1)) \text{ and}\\ &\mathrm{Ext}_{P_3}(F_p, F_p) \otimes \mathrm{Ext}_{P(2)}(F_p, M(p-1)) \Rightarrow \mathrm{Ext}_{P(3)}(F_p, M(p-1))\end{aligned}$$

(*cf.* [5, A1.3.14]) to investigate $\mathrm{Ext}_{P(3)}(F_p, M(p-1)) = F^{s,t}X(p-1)$ associated to the extension of Hopf algebras

$$P_1 \longrightarrow P(2) \longrightarrow P_2 \text{ and } P(2) \longrightarrow P(3) \longrightarrow P_3.$$

Here we recall [2, §2] that

$$\begin{aligned}(2.2)\qquad &\mathrm{Ext}_{P_k}(F_p, F_p) = E(h_{k,i} \mid i \geq 0) \otimes F_p[b_{k,i} \mid i \geq 0] \text{ and}\\ &\mathrm{Ext}_{P_1}(F_p, M(p-1)) = E(h_{1,i} \mid i \geq 1) \otimes F_p[b_{1,i} \mid i \geq 1],\end{aligned}$$

where bidegrees of the elements are:

$$\| h_{k,i} \| = (1, 2p^{i+k} - 2p^k) \text{ and } \| b_{k,i} \| = (2, 2p^{i+k+1} - 2p^{k+1}).$$

Furthermore the differentials of the Cartan-Eilenberg spectral sequences are given as follows:

$$\begin{aligned}&d_2 h_{2,0} = 0\\ &d_2 h_{2,i} = -h_{1,i}h_{1,i+1} \text{ for } i > 0\\ &d_3 b_{2,0} = -b_{1,1}h_{1,1} \text{ and}\end{aligned}$$

$$d_2 h_{3,0} = -h_{2,0}h_{1,2}.$$

These imply

LEMMA 2.3. *Let s and t be integers with $t-s=(p^3+3p+1)q-2$ or $(p^3+3p+2)q-3$. Then*

$$F^{s,t}X(p-1)=0.$$

PROOF. We first suppose that $t-s=(p^3+3p+2)q-3$. By degree reason, it is sufficient to consider the elements of the form

$$b_{1,1}^a b_{2,0}^b h_{1,1}^c h_{1,2}^d h_{2,0}^e h_{2,1}^f h_{3,0}^g, \tag{2.4}$$

where a and b are non-negative integers and b, c, d, e, f and g are 0 or 1. We also see that $(s,t)=(3,(p^3+3p+2)q)$ or $(2p+1,(p^3+3p+3)q)$. It is easy to see that there is no element with bidegree $(3,(p^3+3p+2)q)$. Since the element of the form (2.4) has degree 0, q or $2q$ modulo pq, there is no element with bedegree $(2p+1,(p^3+3p+3)q)$, either. Therefore we have the lemma for this case.

Turn now to the case $t-s=(p^3+3p+1)q-2$. Evaluate the bidegrees of the generators of the E_2-terms of the Cartan-Eilenberg spectral sequences, and we dig out the two elements represented by

$$b_{1,1}^{p-2}b_{2,0}h_{2,0}h_{3,0} \text{ and } b_{1,1}^{p-2}h_{1,1}h_{1,2}h_{2,0}h_{3,0}$$

with total degree $t-s=(p^3+3p+1)q-2$ in $F^{s,t}X(p-1)$. On the other hand, we obtain

$$d_3 b_{1,1}^{p-2}b_{2,0}h_{2,0} = -b_{1,1}^{p-1}h_{1,1}h_{2,0}\neq 0, \text{ and}$$

$$d_2 b_{1,1}^{p-2}h_{2,1}h_{2,0} = b_{1,1}^{p-2}h_{1,1}h_{1,2}h_{2,0}$$

in the E_3-term of the first of the Cartan-Eilenberg spectral sequence. Thus we see that the element $b_{1,1}^{p-2}b_{2,0}h_{2,0}$ dies and the element $b_{1,1}^{p-2}h_{1,1}h_{2,0}$ is killed in the first Cartan-Eilenberg spectral sequence, which implies that there is no element in $F^{2p,(p^3+3p+2)q}X(p-1)$ as desired. q.e.d.

We now recall [8] the computation of $\mathrm{Ext}_P^{s,t}(F_p,F_p)$ which is $F^{s,t}X(0)$ for $t-s\le 2p^4-2$:

(2.5) *Let $E_2=F_p[b_{ij}\mid i,j\ge 0]\otimes H^{*,*}(U(L))\Rightarrow \mathrm{Ext}_P^{*,*}(F_p,F_p)$ be the May spectral sequence. In the E_2-term, $H^{*,*}(U(L))$ is additively generated by*

$$1,\ h_0,\ h_1,\ g_0,\ k_0,\ k_0h_0,\ l_4,\ l_4h_0,\ l_4h_1,\ l_4g_0,\ l_4k_0,\ l_4k_0h_0,$$

$$h_3,\ h_3h_0,\ h_3h_1,\ h_3g_0, h_3k_0,\ h_3k_0h_0,$$

$$h_2,\ h_2h_0,\ g_1,\ l_1,\ l_2,\ l_1h_1,\ k_1,\ l_3,\ k_1h_1,\ l_1h_2,\ m_1,\ m_1h_0$$

for $t-s\le q(4)$. Here each element has bidegree:

$\|1\| = (0, 0)$, $\|h_0\| = (1, q)$, $\|h_1\| = (1, pq)$,

$\|g_0\| = (2, (p+2)q)$, $\|k_0\| = (2, (2p+1)q)$, $\|k_0h_0\| = (3, (2p+2)q)$,

$\|l_4\| = (3, (3p^2+2p+1)q)$, $\|l_4h_0\| = (4, (3p^2+2p+2)q)$,

$\|l_4h_1\| = (4, (3p^2+3p+1)q)$, $\|l_4g_0\| = (5, (3p^2+3p+3)q)$,

$\|l_4k_0\| = (5, (3p^2+4p+2)q)$, $\|l_4k_0h_0\| = (6, (3p^2+4p+3)q)$,

$\|h_3\| = (1, p^3q)$, $\|h_3h_0\| = (2, (p^3+1)q)$, $\|h_3h_1\| = (2, (p^3+p)q)$,

$\|h_3g_0\| = (3, (p^3+p+2)q)$, $\|h_3k_0\| = (3, (p^3+2p+1)q)$,

$\|h_3k_0h_0\| = (4, (p^3+2p+2)q)$,

$\|h_2\| = (1, p^2q)$, $\|h_2h_0\| = (2, (p^2+1)q)$, $\|g_1\| = (2, (p^2+2p)q)$,

$\|l_1\| = (3, (p^2+2p+3)q)$, $\|l_2\| = (3, (p^2+3p+1)q)$, $\|l_1h_1\| = (4, (p^2+3p+3)q)$,

$\|k_1\| = (2, (2p^2+p)q)$, $\|l_3\| = (3, (2p^2+p+2)q)$, $\|k_1h_1\| = (3, (2p^2+2p)q)$,

$\|l_1h_2\| = (4, (2p^2+2p+3)q)$, $\|m_1\| = (4, (2p^2+4p+2)q)$,

$\|m_1h_0\| = (5, (2p^2+4p+3)q)$,

and

$$\|b_{i,j}\| = (2, 2p^{j+1}(p^i-1)) = (2, p^{j+1}(p^{i-1}+\cdots+1)q)$$

Evaluating the degrees of generators shows

LEMMA 2.6. *Let s and t be integers such that $s \le 2p-1$ and $t-s = (p^3+2p+2)q-1$. Then*

$$E^{s,t}V(3) = F^{s,t}X(0) = 0.$$

PROOF. In this proof, we abbreviate $F^{s,t}X(0)$ to $F^{s,t}$. Since $F^{s,t} = 0$ unless $t \equiv 0$ (q), we assume that $t \equiv 0$ (q). Then we have $s \equiv 1$ (q) by the assumption of the lemma, which implies $s = 1$ or $2p-1$ since $s \le 2p-1$. By Toda's computation (2.5), we easily get that $F^{1,t} = 0$.

Now suppose further that $s = 2p-1 = q+1$. Then $t = (p^3+2p+3)q$, and we dig out elements of the form

$$b_x x \in F^{2p-1,(p^3+2p+3)q},$$

where b_x is a multiple of $b_{i,j}$'s and x is one of the elements

$$l_4g_0,\ l_1,\ m_1h_0,$$

by noticing that the prime p divides the inner degrees of $b_{i,j}$'s and 2 is the homology degrees of $b_{i,j}$'s. The bidegrees of b_x are

$$(2p-6,\ (p^3-3p^2-p)q),\ (2p-4,\ (p^3-p^2)q) \text{ and } (2p-6,\ (p^3-2p^2-2p)q)$$

for $x = l_4 g_0$, l_1 and $m_1 h_0$, respectively. By the degree reason, we can put

$$b_x = b_{1,0}^a b_{1,1}^b b_{2,0}^c$$

with bidegree $(2(a+b+c), ((b+c)p^2+(a+c)p)q)$. Consider the equations $a+b+c = p-3$ and $((b+c)p^2+(a+c)p)q = (p^3-3p^2-p)q$. Then $b+c = p-3$ and so

$$c = a(p-1)-1,$$

which does not satisfy the inequality $0 \le c \le p-3$ obtained from the first equation.

For the cases $x = l_1$ and $m_1 h_0$, we similarly deduce

$$c = a(p-1)+p \text{ and } c = a(p-1)+p-2,$$

which contradict to

$$0 \le c \le p-2 \text{ and } 0 \le c \le p-3,$$

respectively. Therefore we have no element of the form $b_x x$, and so we have the lemma. q.e.d.

§3. The second obstruction

In the same manner as the proof of Lemma 2.6, Toda's computation (2.5) gives the obstructions

$$\xi_1 = b_{2,0}^{p-3} l_4 k_0,\ \xi_2 = b_{1,0}^{p-2} b_{1,1}^{p-2} b_{2,0} l_1,\ \xi_3 = b_{1,0}^{p^2-2p-2} b_{2,0}^3 h_1,$$

$$\xi_4 = b_{1,0}^{p^2-2p-2} b_{2,0} k_1 l_1 \text{ and } \xi_5 = b_{1,0}^{p^2-p-2} b_{2,0} l_2$$

in the E_2-term $E^{*,*}X(0) = \mathrm{Ext}_{BP_*BP}^{*,*}(BP_*,\ BP_*/I_4)$ of the Adams-Novikov spectral sequence for $V(3)$. In this section we will show that $\xi_2 = 0$.

Take the smash product $V(3)$ and the cofibering $\Sigma^{pq-2}S \xrightarrow{\beta_1} S \to C$ obtained by the generator β_1 of the homotopy group $\pi_{pq-2}S$ of the sphere spectrum S, and we obtain the cofibering

$$\Sigma^{pq-2}V(3) \xrightarrow{b} V(3) \xrightarrow{iw} W,$$

whose first map induces the map

$$b_{1,0}\colon E^{s,t}V(3) \longrightarrow E^{s+2,t+pq}V(3)$$

defined as multiplication with $b_{1,0}$ (*cf.* [6]). Furthermore we have

(3.1) [6] *Let s and t be integers such that $s \le 2p+3$ and $t-s \le 2p^4-2$ and $s < 2p+2$ if $t-s \le (p^3+p^2-1)q-4$. Then*

$$E^{s,t}W = 0.$$

PROOF OF THEOREM. By [6], we see that $d_r v_4 = 0$ for $r \le 2q$. Let ξ denote the element $b_{1,1}^{p-2} b_{2,0} l_1$ which would belong to

$$E^{s,t}V(3) = F^{s,t}X(0)$$

for $(s, t) = (2p + 1, (p^3 + 3p + 3)q)$. Then $\xi_2 = b_{1,0}^{p-2}\xi$. Since $b_{1,0}^j\xi$ has homology degree greater than $2p + 2$ for $j > 0$, the induced map i_{W*} carries it to zero by (3.1). Thus we have $i_W x = 0$, if $b_{1,0}^j\xi$ survives to the homotopy element x. Therefore we find an element y in $\pi_* V(3)$ such that $b_* y = x$. Note here that the filtration of y is less than that of x. Comparing degrees of elements in (2.5) shows that $y = b_{1,0}^{j-1}\xi$ as long as $j > 0$. Consider the diagram

$$\begin{tikzcd}
 & \Sigma^{-1}X(p-1) \arrow[r, equal] \arrow[d] & \Sigma^{-1}X(p-1) \arrow[d] & \\
\Sigma^{pq-2}V(3) \arrow[r] \arrow[d, equal] & \Sigma^{q-1}X(p-2) \arrow[r] \arrow[d, "1\wedge g_{p-2}"] & \Sigma^{q-1}X(p-1) \arrow[r] \arrow[d] & \Sigma^{pq-1}V(3) \arrow[d, equal] \\
\Sigma^{pq-2}V(3) \arrow[r, "b"] & V(3) \arrow[r] \arrow[d, "1\wedge i"] & W \arrow[r] \arrow[d] & \Sigma^{pq-1}V(3) \\
 & X(p-1) \arrow[r, equal] & X(p-1) &
\end{tikzcd}$$

given in [6]. Applying Ext to the second vertical sequence shows that $E^{s,t}W = 0$ for s and t with $t - s = (p^3 + 3p + 2)q - 3$ by Lemma 2.3. This shows that $i_{W*}\xi = 0$. On the other hand, Lemma 2.6 implies that $\xi = 0$ if and only if $i_{W*}\xi = 0$. Hence we obtain $\xi = 0$, and so $\xi_2 = 0$ which implies $d_{2q+1}v_4 = 0$. The next obstruction stays in the module at homology degree $2p^2 - 4q + 3$, and so we have the desired results.

q.e.d.

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